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# A Numerical Approach to 3-D Inverse Scattering Problems

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**Abstract**—A new approach to the numerical solution of the 3-D inverse scattering problems is given.

**Keywords**—Inverse scattering, Variational method, Optimization, Parameter fitting.

## 1. INTRODUCTION

Let

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } R^3, \quad k = \text{const} > 0, \tag{1}$$

$$u = \exp(ik\alpha \cdot x) + A(\alpha', \alpha, k)\frac{\exp(ikr)}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \frac{x}{r} = \alpha'. \tag{2}$$

Here $\alpha \in S^2$, $S^2$ is the unit sphere, $A = A(\alpha', \alpha, k)$ is called the scattering amplitude. Let us assume that $q(x) \in Q_a := \{q : q = \overline{q}, q \in L^2(\mathbb{R}^3), q = 0 \text{ for } |x| > a\}$, $a > 0$ is an arbitrary large fixed number. The inverse scattering problem consists of finding $q(x)$ from the given $A(\alpha', \alpha, k)$.

Uniqueness of the solution to this problem for the case when $k > 0$ is fixed and $A(\alpha', \alpha, k)$ is known for all $\alpha', \alpha \in S^2$ is proved by the author [1]. A numerical method for solving this inverse problem is given in [2], where the error estimates are obtained and stability of the inversion of the noisy data is studied.

The inversion method given in [2] is exact and its convergence is proved, but the method is not simple.

The purpose of this paper is to give a new approach to the numerical solution of the inverse scattering problem. Conceptually, this approach is simple.

In Section 2, this approach is described.

## 2. PRELIMINARY REMARKS

The starting point is the well-known formula

$$-4\pi A(\alpha', \alpha, k) = \int \exp(-ik\alpha' \cdot y)q(y)u(x, \alpha, k)\,dx, \quad \int := \int_{|x|\le a}, \tag{3}$$

(see e.g., [3] for a brief account of the scattering theory). Let us denote

$$q(x)u(x, \alpha, k) := f(x; \alpha, k). \tag{4}$$

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Let us fix $\alpha \in S^2$. Write (3) as

$$-4\pi A(\alpha', \alpha, k) = \int \exp(-ik\alpha' \cdot x) f(x, \alpha, k)\, dx. \tag{5}$$

This is an integral equation for $f(x, \alpha, k)$. For fixed $\alpha \in S^2$ and $k > 0$, the operator in (5) maps $L^2(B_a)$ into $L^2(S^2)$. The data $-4\pi A(\alpha', \alpha, k)$ is a function of $\alpha'$ for fixed $\alpha$ and $k$. For exact data equation (5) has a solution. In fact, this equation has infinitely many solutions. Suppose $f(x, \alpha, k)$ is a solution of (5). Calculate

$$w := \exp(ik\alpha \cdot x) - \int \frac{\exp(ik|x-y|)}{4\pi|x-y|} f(y, \alpha, k)\, dy \tag{6}$$

and

$$q(x, \alpha, k) := \frac{f(x, \alpha, k)}{w(x, \alpha, k)}. \tag{7}$$

If the function (7) does not depend on $\alpha$ (and $k$), then $q(x, \alpha, k) = q(x)$ is the desired potential, which is unique by the above mentioned uniqueness theorem. The numerical problem is to choose such a solution to (5) that function (7) does not depend on $\alpha$ and $k$. Let us explain (6) and (7). If we find the solution to (4) of the form $f = q(x)u(x, \alpha, k)$, then the following equation holds:

$$(\nabla^2 + k^2)u = f \tag{8}$$

by virtue of equation (1). One can find $u(x, \alpha, k)$ from (8) by formula (6) and the potential $q(x)$ by the formula: $q(x) = f/u(x, \alpha, k)$. This is the justification of the formulas (6) and (7). The numerical problem of finding the (unique) solution to (5) with the following property: the function $q(x, \alpha, k)$ defined by formula (6) does not depend on $\alpha$ and $k$, is a difficult problem.

One may try to approximate such a solution by *minimizing* the function

$$\left| \frac{f(x, \alpha, k)}{w(x, \alpha, k)} - \frac{f(x, \alpha_1, k)}{w(x, \alpha_1, k)} \right| = \min. \tag{9}$$

Here, $\alpha_1$ is an arbitrary fixed unit vector, $k > 0$ is assumed to be fixed, and the minimization in (9) is taken over all the solutions to equation (5). One can look for solutions to (5) of the form

$$f(x) = \sum_{\ell=0}^{\infty} f_\ell(r, \alpha, k) Y_\ell(x^0), \quad x^0 := \frac{x}{|x|}, \quad r := |x|, \tag{10}$$

where $Y_\ell(x^0)$ are the orthonormalized in $L^2(S^2)$ spherical harmonics, $\sum_{\ell=0}^{\infty} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}$, $Y_\ell = Y_{\ell m}$. Expanding both sides of (5) in spherical harmonics (in the variable $\alpha'$) and equating the coefficients in front of $Y_\ell(\alpha')$, one gets

$$-A_\ell(\alpha, k) = (-i)^\ell \int j_\ell(k|x|) \overline{Y_\ell(x^0)} f(x, \alpha, k)\, dx, \quad \ell = 0, 1, 2, \ldots, \tag{11}$$

where $A_\ell(\alpha, k)$ are the Fourier coefficients of $A(\alpha', \alpha, k)$. Using (10) one gets from (11):

$$-(-i)^{-\ell} A_\ell(\alpha, k) = \int_0^a dr r^2 j_\ell(kr) f_\ell(r, \alpha, k), \quad \ell = 0, 1, 2, \ldots \tag{12}$$

If $\alpha \in S^2$ and $k > 0$ are fixed, then equation (12) has many solutions: if $f_\ell := f_\ell(r, \alpha, k)$ is a solution then $f_\ell + h_\ell$, where $h_\ell = h_\ell(r, \alpha, k)$ is an arbitrary function orthogonal in $L^2(0, a)$ to $r^2 j_\ell(kr)$, is also a solution to (12).

One can minimize the expression in (9) with respect to $h_\ell$ in order to find $q(x)$. This minimization step is the major difficulty in the suggested approach.

# 3. A NUMERICAL APPROACH TO INVERSE SCATTERING PROBLEMS

Let us give a variational approach to solving inverse scattering problems (ISP). We construct a functional whose unique global minimizer yields the solution to ISP. We prove convergence of minimizing sequences to the solution of ISP.

1. Inverse potential scattering (IPS). We start with the equation (5) and the equation

$$f = q(u_0 - Gf), \tag{13}$$

where

$$u_0 := \exp(ik\alpha \cdot x), \quad Gf := \int \frac{\exp(ik|x-y|)}{4\pi|x-y|} f(y)\, dy. \tag{14}$$

Consider equations (5) and (13) as a system of two equations for two unknown functions $q(x) \in Q_a$ and $f(x, \alpha, k)$. Equation (5) is a linear Fredholm first kind equation for $f$ and (13) is a nonlinear equation.

Lemma 1. *The system (5), (13) has at most one solution $\{q, f\}$, with $q \in Q_a$.*

Proof. Assume that $\{q_j, f_j\}$, $j = 1, 2$, are solutions to (5), (13), $q_j \in Q_a$. Define $u_j := u_0 - Gf_j$. Clearly $(\nabla^2 + 1)u_j = f_j$ and $f_j = q_j u_j$ by (13). Therefore, $u_j$ solve equations (1), (2) with $q = q_j$ and have the same scattering amplitudes: $A_1(\alpha', \alpha) = A_2(\alpha', \alpha)$. By the author's uniqueness theorem [1, p. 64], it follows that $q_1(x) = q_2(x)$. Therefore, $u_1 = u_2$ and $f_1 = f_2$. Lemma 1 is proved. ∎

Define the functional

$$I(p, v) := \left\| 4\pi A(\alpha', \alpha) + \int \exp(-i\alpha' \cdot x) p(x)[u_0 + v]dx \right\|_1 + \left\| p(x)[v + Gp(u_0 + v)] \right\|_2. \tag{15}$$

Here, $u_0$ is defined in (14), $p(x) \in L^2(B_a)$, $v = v(x, \alpha) \in L^2(B_a \times S^2)$, and

$$\|\cdot\|_1 := \|\cdot\|_{L^2(S^2 \times S^2)}, \quad \|\cdot\|_2 := \|\cdot\|_{L^2(B_a \times S^2)}. \tag{16}$$

If $I(p, v) = 0$ then the pair $\{p, f\}$, $f := p(u_0 + v)$, solves the system (5), (13). Indeed, if $I(p, v) = 0$ then clearly equation (5) holds and $p[v + Gp(u_0 + v)] = 0$. The last equation one can rewrite as equation (13) with $p(x)$ in place of $q$ and $f = p(u_0 + v)$. By Lemma 1, the system of equations (5), (13) has at most one solution. Therefore, $p(x) = q(x)$ and $v(x, \alpha) = v_q := u_q - u_0$, where $u_q := u_q(x, \alpha)$ is the scattering solution corresponding to $q(x)$. We have proved the following:

Lemma 2. *The global minimizer of the functional $I(p, v)$ is unique and is the pair $\{q, v_q\}$, $q = q(x)$ is the solution to ISP and $v_q := u_q - u_0$, where $u_q(x, \alpha)$ is the scattering solution corresponding to the potential $q(x)$. One has $I(q, v_q) = 0$.*

Consider the problem

$$I(p, v) = \min, \tag{17}$$

where the minimization is taken over $p \in L^2(B_a)$ and $v \in L^2(B_a \times S^2)$. Assume that the global minimizer $\{q, v_q\}$ of (17) lies in a compact set $K$ of the space $L^2(B_a) \times L^2(B_a \times S^2)$. Denote $h := \{p, v\}$. Introduce a norm $\|h\|$ such that $I(h)$ is continuous with respect to this norm, that is, $I(h_n) \to I(h)$ if $\|h_n - h\| \to 0$. For instance, one can take as $K$ the set of $h$ such that $\|h\| < c$, where $c > 0$ is a constant and $\|h\| = \|p(x)\|_{C^\ell(B_a)} + \|v\|_{C^\ell(B_a \times S^2)}$, $\ell > 0$.

Choose an arbitrary minimizing sequence $h_n \in K$ for problem (17) assuming that the data $A(\alpha', \alpha)$ are exact: $I(h_n) \to 0$. Since $K$ is compact one can assume that $h_n \to h$ in $C(B_a) \times C(B_a \times S^2)$, and, by the continuity of $I(p, v)$ with respect to $C(B_a) \times C(B_a \times S^2)$ norm one has $I(h) = 0$. By Lemma 2 it follows that $h = \{q, v_q\}$. We have proved the following theorem.

THEOREM 1. *Assume $q \in C^1 \cap Q_a$, $\|q\|_{C^1(B_a)} \le c$. Then any minimizing sequence $h_n \in K$, such that $I(h_n) \to 0$ as $n \to \infty$, converges uniformly to $h = \{q, v_q\}$, where $q = q(x)$ is the solution to ISP.*

2. Consider IGS (inverse geophysical scattering). Let us use the same ideas for solving IGP, inverse geophysical problems. Let

$$Lu := \left[\nabla^2 + k^2 + k^2 v(x)\right] u(x, y, k) = -\delta(x - y) \quad \text{in } \mathbb{R}^3, \tag{18}$$

$u$ satisfies the radiation condition, $v(x) \in Q_a$, $v(x) = 0$ for $x_3 \ge 0$, $P := \{x : x_3 = 0\}$, $B^- := B_a \cap \mathbb{R}^3_-$, $\mathbb{R}^3_- := \{x : x_3 < 0\}$. The surface data are $u(\hat{x}, \hat{y}, k)$, $\forall \hat{x}, \hat{y} \in P$, $k > 0$ is fixed. Consider the problem:

$$\mathcal{J} := \left\|f - vg - k^2 vGf\right\|_3 + \left\|u(\hat{x}, \hat{y}) - g - Gf\right\|_4 = \min, \tag{19}$$

where $f := v(z)u(z, \hat{y})$,

$$\begin{aligned}
&\|\cdot\|_3 = \|\cdot\|_{L^2(B^-)\times L^2(P;(1+|\hat{x}|^2)^{-1})}, \quad \hat{x} := (x_1, x_2),\\
&\|\cdot\|_4 = \|\cdot\|_{L^2\left(P\times P;\,(1+|\hat{x}|^2)^{-1}(1+|\hat{y}|^2)^{-1}\right)},\\
&g = \frac{\exp\left(ik|\hat{x} - \hat{y}|\right)}{4\pi\,|\hat{x} - \hat{y}|}, \quad Gf = \int_{B^-} g(x, z)\, f(z)\, dz.
\end{aligned}$$

If $f(z, \hat{y}) = v(z)u(z, \hat{y}, k)$, where $u$ solves (18), then $\mathcal{J}(v, f) = 0$. Minimization in (19) is taken over the functions $v \in L^2(B^-)$ and $f \in L^2(B^-) \times L^2(P; (1 + |\hat{x}|^2)^{-1})$. Let $K$ be a compactum in the space of the elements $h = \{v, f\}$, $\|h\| = \|v\|_{L^2(B^-)} + \|f\|_3$, and assume that the solution $\{v, f\}$ to the IGP lies in $K$. Let $h_n$ be any minimizing sequence: $\mathcal{J}(h_n) \to 0$ as $n \to \infty$, which belongs to $K$. Then $h_n \to h$ since $K$ is a compactum. Assume that $\mathcal{J}$ is continuous with respect to this convergence. Then the following theorem holds.

THEOREM 2. *Under the above assumptions the limit $h = \{v, f\}$ of the minimizing sequence $h_n$ yields the solution to IGP. The functional $\mathcal{J}$ has a unique global minimum which is attained only at the solution to IGP. Any minimizing sequence $h_n$ such that $\mathcal{J}(h_n) \to 0$, $h_n \to h$, $\mathcal{J}(h_n) \to \mathcal{J}(h)$ converges to the solution to IGP.*

Proof of Theorem 2 is similar to that of Theorem 1.

3. The same ideas can be used for inverse obstacle scattering (IOS). Let $D$ be a bounded domain with a sufficiently smooth boundary $\Gamma$, for example, $C^{1,\lambda}$ boundary or even Lipschitz boundary. The IOS problem consists of finding $\Gamma$ given the scattering amplitude $A(\alpha', \alpha, k)$ for all $\alpha', \alpha \in S^2$ and a fixed $k > 0$. One assumes the Dirichlet boundary condition on $\Gamma$ (the Neumann condition can be assumed as well).

The governing equations are [1]:

$$-4\pi A(\alpha', \alpha) = \int_\Gamma \exp(-ik\alpha' \cdot s) u_N(s)\, ds \tag{20}$$

$$\int_\Gamma g(t, s) u_N(s)\, ds = \exp(ik\alpha \cdot t), \qquad t \in \Gamma, \quad g = \frac{\exp(ik|t - s|)}{4\pi|t - s|}. \tag{20'}$$

Assume that $\Gamma$ can be given by the equation $r = r(\theta)$, $\theta \in S^2$, $ds = H(\theta)\, d\theta$, $u_N(r(\theta)\theta; \alpha)H(\theta) := f(\theta; \alpha)$. Then (20), (20′) can be written as

$$0 = 4\pi A(\alpha', \alpha) + \int_{S^2} \exp[-ik\alpha' \cdot \theta r(\theta)] f(\theta; \alpha)\, d\theta, \tag{21}$$

$$0 = \int_{S^2} g(\beta r(\beta), \theta r(\theta)) f(\theta; \alpha)\, d\theta - \exp[ik\alpha \cdot \beta r(\beta)]. \tag{22}$$

Consider the problem:

$$F(h) := F(r(\theta), f(\theta; \alpha)) = \min, \quad h := \{r(\theta), f(\theta; \alpha)\}, \tag{23}$$

where

$$\begin{aligned} F := &\left\| 4\pi A(\alpha', \alpha) + \int_{S^2} \exp[-ik\alpha' \cdot \theta r(\theta)] f(\theta; \alpha)\, d\theta \right\|_5 \\ &+ \left\| \int_{S^2} g(\beta r(\beta), \theta r(\theta)) f(\theta; \alpha)\, d\theta - \exp[ik\alpha \cdot \beta r(\beta)] \right\|_5 \end{aligned} \tag{24}$$

and where $\|\cdot\|_5 = \|\cdot\|_{L^2(S^2 \times S^2)}$.

Let the solution $\{r(\theta), f(\theta; \alpha)\}$ belong to a suitable compactum $K$ in $L^2(S^2) \times L^2(S^2 \times S^2)$. The functional $F(h)$ has global minimum value zero. The only global minimizer $h$ of $F$ is the pair $\{r(\theta), f(\theta; \alpha)\}$, where $r = r(\theta)$ is the equation of $\partial D$. Lemma similar to Lemma 1 holds for the system of equations (21), (22). Its proof is similar to the proof of Lemma 1. Let $h_n \to h$ imply $F(h_n) \to F(h)$. Then we say that $h_n$ converges properly. We prove the following theorem.

THEOREM 3. *If $h_n \in K$, $h_n \to h$, $F(h_n) \to 0$, then the pair $h = \{r(\theta), f(\theta; \alpha)\}$ yields the unique solution of the inverse scattering problem for the obstacle $D$.*

The proof of Theorem 3 is similar to that of Theorem 1.

SUMMARY. For IPS, IGS, and IOS, we have proposed the functionals with the properties:

(1) *minimization of these functionals is equivalent to solving the inverse scattering problems: the functionals have a unique global minimum zero and a unique global minimizer which is the solution to the inverse scattering problem;*

(2) *any properly convergent minimizing sequence for these functionals converges to the (unique) solution to the inverse scattering problem.*

The ideas of this paper can be used for solving inverse scattering problems for Maxwell's equations and elasticity theory.

## REFERENCES

1. A.G. Ramm, *Multidimensional Inverse Scattering Problems*, Longman, New York, (1992), (expanded Russian edition, Mir, Moscow, 1993).
2. A.G. Ramm, Stability estimates in inverse scattering, *Acta Appl. Math* **28**, **N1**, 1–42 (1992).
3. A.G. Ramm, *Random Fields Estimation Theory*, Longman, New York, (1990).